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# Vortex State of d-Wave Superconductors in the Ginzburg-Landau Energy

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## Abstract

We find a minimizer of a reduced form of the Ginzburg-Landau free energy for  $d$ -wave superconductors having distinct degree-one vortices. For a single vortex in the vortex core, we analytically recover the vortex structure with fourfold symmetry.

## 1 Introduction

In the 1910's, low-temperature superconductivity was observed on metals and alloys (cf. [9]). Recently, high-temperature superconductivity has been found on some copper-oxide superconductors (cf. [12]). The vortex state of high-temperature superconductors is different from the vortex state of low-temperature superconductors. When the applied magnetic field is close to the lower critical field  $H_{c1}$ , the single vortex is expected to be symmetric in low-temperature superconductors but it may be asymmetric (fourfold symmetric) in high-temperature superconductors (cf. [8], [31]). Moreover, as the applied magnetic field is close to the upper critical field  $H_{c2}$ , Abrikosov type vortex lattices are expected to be triangular in low-temperature superconductors but they may be rectangular in high-temperature superconductors (cf. [1], [8], [27], [30], [31] etc).

To distinguish low-temperature and high-temperature superconductivity, an  $s$ -wave and a  $d$ -wave order parameter were introduced (cf. [13], [21]). Soininen et al. (cf. [3], [28]) introduced the Ginzburg-Landau free energy with an  $s$ -wave and a  $d$ -wave order parameter. Ren et al. (cf. [24], [25]) present a microscopic derivation of the Ginzburg-Landau equations from the Gor'kov equations by using the finite temperature Green's-function approximation method. From [31], we learned the two fields Ginzburg-Landau free energy is given by:

$$G(\Psi_s, \Psi_d, A) = \int_{\mathbb{R}^2} \kappa^2 |\operatorname{curl} A - H|^2 + \alpha_s(T) |\Psi_s|^2 + \frac{1}{2} (1 - |\Psi_d|^2)^2 + \frac{4}{3} |\Psi_s|^4 + \frac{8}{3} |\Psi_s|^2 |\Psi_d|^2 + \frac{2}{3} (\Psi_s^2 \Psi_d^{*2} + \Psi_d^2 \Psi_s^{*2}) + 2 |\Pi \Psi_s|^2 + |\Pi \Psi_d|^2 + \{ \Pi_x \Psi_s \Pi_x^* \Psi_d^* - \Pi_y \Psi_s \Pi_y^* \Psi_d^* + \text{H.C.} \}, \quad (1.1)$$

where  $\Psi_s$  is the  $s$ -wave order parameter,  $\Psi_d$  is the  $d$ -wave order parameter and  $A$  is the vector-valued magnetic potential,  $\Pi = i\nabla - A$ ,  $H$  is a constant applied magnetic field,  $\kappa$  is the Ginzburg-Landau parameter and

$$\alpha_s(T) = C_s/(1 - T/T_c). \quad (1.2)$$

Here  $C_s$  is a positive constant,  $T$  is the current temperature and  $T_c$  is the  $d$ -wave transition temperature.

As the current temperature  $T$  is close to  $T_c$ , Franz et al. [8] observed that in a predominantly  $d$ -wave superconductor, the  $s$ -wave component is generically very small. They also provided approximation formulas for the order parameters  $\Psi_d$  and  $\Psi_s$  as follows:

$$|\Psi_s| \ll |\Psi_d|, \quad |\nabla \Psi_s| \ll |\nabla \Psi_d| \quad \text{as } T \rightarrow T_c. \quad (1.3)$$

Affleck et al. [1] obtained the leading order in  $(1 - T/T_c)$  as

$$\Psi_s = \xi (\prod_x^2 - \prod_y^2) \Psi_d, \quad (1.4)$$

where  $\xi$  is a parameter satisfying that  $\xi \rightarrow 0$  as  $T \rightarrow T_c$ . In [7], Du derived (1.4) by the formal asymptotic analysis.

We learned from [5] and [6] that it is reasonable to ignore the magnetic field in strongly type II superconductors when the applied magnetic field is close to  $H_{c1}$  and  $T \rightarrow T_c$ . Hence it is valuable to study the two fields Ginzburg-Landau model (1.1) without the magnetic field (i.e.  $A, H \equiv 0$ ). Moreover, Rosenstein et al. [6] took (1.3) and (1.4) into (1.1) and modified the free energy (1.1) as follows:

$$G(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 dx dy, \quad (1.5)$$

where  $\square = \partial_x^2 - \partial_y^2$  and  $\beta$  is a parameter satisfying that  $\beta \rightarrow 0$  as  $T \rightarrow T_c$ . Here we have ignored the magnetic field (i.e.  $A, H \equiv 0$ ) for strongly type II superconductors.

It is hard to find the minimizer of (1.5) by the standard direct method. Suppose that  $\Psi_d \in H^2(\mathbb{R}^2; \mathbb{C})$  is a minimizer of (1.5) over  $H^2(\mathbb{R}^2; \mathbb{C})$ . Then it is easy to check that

$$\begin{aligned} G(\Psi_d + v) = & G(\Psi_d) + \int_{\mathbb{R}^2} |\nabla v|^2 - (1 - |\Psi_d|^2)|v|^2 + 2(\Psi_d \cdot v)^2 \\ & + \int_{\mathbb{R}^2} 2|v|^2(\Psi_d \cdot v) + \frac{1}{2}|v|^4 + \beta |\square v|^2, \end{aligned} \quad (1.6)$$

for any test function  $v \in C_0^\infty(\mathbb{R}^2)$ . Hereafter,  $z_1 \cdot z_2 = \frac{1}{2}(\bar{z}_1 z_2 + z_1 \bar{z}_2)$  for all  $z_1, z_2 \in \mathbb{C}$ . Let  $v_n(z) = \delta_n v_0(z) \sin[\delta_n^{-2/3}(x+y)]$  for  $z = x + iy \in \mathbb{C} \cong \mathbb{R}^2$ , where  $v_0$  is a test function with a nonempty compact support and  $\{\delta_n\}$  is a sequence of positive numbers such that  $\delta_n \rightarrow 0$  as  $n \rightarrow \infty$ . Here we use the fact that the complex plane  $\mathbb{C}$  is isomorphic to  $\mathbb{R}^2$ . Now, we replace  $v$  in (1.6) by  $v_n$  and we obtain that  $G(\Psi_d + v_n) \rightarrow G(\Psi_d)$  but  $\|\Psi_d + v_n\|_{H^2} \rightarrow \infty$  as  $n \rightarrow \infty$ . Hence  $\Psi_d + v_n$ 's form a minimizing sequence but  $\Psi_d + v_n$ 's have no converging subsequence

even weakly converging subsequences in  $H_{loc}^2(\mathbb{R}^2; \mathbb{C})$ . Thus the free energy (1.5) has a defect on minimization.

From [30], we learned a Ginzburg-Landau energy functional (without the magnetic field) as follows:

$$E(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \eta(|\partial_x^2 \Psi_d|^2 + |\partial_y^2 \Psi_d|^2) dx dy, \quad (1.7)$$

where  $\eta$  is a constant depending on the current temperature  $T$ . The term  $|\partial_x^2 \Psi_d|^2 + |\partial_y^2 \Psi_d|^2$  breaks the circular symmetry and accounts for the square symmetry. Furthermore, Park and Huse [22] introduced a more generalized Ginzburg-Landau free energy (without the magnetic field) for  $d$ -wave superconductors as follows:

$$F(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \gamma_1 |\Delta \Psi_d|^2 + \beta_1 (|\square \Psi_d|^2 - 4|\partial_x \partial_y \Psi_d|^2) dx dy, \quad (1.8)$$

where  $\Delta = \partial_x^2 + \partial_y^2$  and  $\beta_1, \gamma_1$  are parameters tending to zero as  $T \rightarrow T_c$ .

Hereafter, we assume that  $|\Psi_d| \rightarrow 1$  and all the derivatives of  $\Psi_d$  decay fast as  $|(x, y)| \rightarrow \infty$ . Such an assumption is consistent with the results in [8] and [31]. Using integration by part, we may transform (1.8) into

$$\tilde{G}(\Psi_d) = \int_{\mathbb{R}^2} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 dx dy, \quad (1.9)$$

where  $\beta, \gamma$  are parameters tending to zero as  $T \rightarrow T_c$ . In this paper, we assume that  $\beta, \gamma > 0$  and  $\beta, \gamma \rightarrow 0$  as  $T \rightarrow T_c$ . In particular, such an assumption includes the case that  $0 < \gamma \ll \beta$  i.e. (1.9) is a small perturbation of (1.5).

In Section 2, we approximate (1.9) by

$$G_\epsilon(\Psi_d) = \int_{\frac{1}{\epsilon}\Omega} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 dx dy, \quad (1.10)$$

where  $0 < \epsilon \ll 1$  is a small parameter,  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^2$  having an interior point at the origin and  $\frac{1}{\epsilon}\Omega = \{(\frac{x}{\epsilon}, \frac{y}{\epsilon}) : (x, y) \in \Omega\}$ . In the rest of this paper, we prove that the minimizer of (1.10) has distinct degree-one vortices in Section 3. In Section 4, we replace  $\frac{1}{\epsilon}\Omega$  in (1.10) by  $B_{R_0}$ , where  $B_{R_0}$  is a disk with radius  $R_0$  and center at the origin. Here  $R_0 > 0$  is a large constant satisfying  $1 \ll R_0 \leq 1/\epsilon$ . Then (1.10) becomes

$$\hat{G}(\Psi_d) = \int_{B_{R_0}} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 dx dy, \quad (1.11)$$

where  $\beta > 0$  is a small parameter as  $T \rightarrow T_c$ ,  $\gamma = C\beta$ , and  $C$  is a positive constant independent of  $\beta$ . We study then the critical point of (1.11) and find out its single vortex structure with fourfold symmetry. The single vortex structure of  $d$ -wave superconductors having fourfold symmetry is well known in physics (cf. [5], [6], [8], [27] and [31]). Here we give a mathematical proof of such a vortex structure.

## 2 Preliminaries

To investigate vortices in  $d$ -wave superconductors, we assume that the order parameter  $\Psi_d$  satisfies  $|\Psi_d| \rightarrow 1$  and all the derivatives of  $\Psi_d$  decay fast as  $|(x, y)| \rightarrow \infty$ . Such an assumption is consistent with the results in [8] and [31]. Hence we may approximate (1.9) by

$$G_\epsilon(\Psi_d) = \int_{\frac{1}{\epsilon}\Omega} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 dx dy, \quad (2.1)$$

where  $0 < \epsilon \ll 1$  is a small parameter,  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^2$  having an interior point at the origin and  $\frac{1}{\epsilon}\Omega = \{(\frac{x}{\epsilon}, \frac{y}{\epsilon}) : (x, y) \in \Omega\}$ . Rescaling the spatial variables  $x, y$  by  $\epsilon$ , (2.1) becomes

$$\hat{G}_\epsilon(\Psi_d) = \int_{\Omega} |\nabla \Psi_d|^2 + \frac{1}{2\epsilon^2}(1 - |\Psi_d|^2)^2 + \delta_\epsilon |\square \Psi_d|^2 + \gamma_\epsilon |\Delta \Psi_d|^2 dx dy, \quad (2.2)$$

where

$$\delta_\epsilon = \beta \epsilon^2 \quad \text{and} \quad \gamma_\epsilon = \gamma \epsilon^2. \quad (2.3)$$

Of course, (2.3) implies that  $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$  as  $\epsilon \rightarrow 0+$ . In Section 2 and 3, we study (2.2) with an assumption that  $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$  as  $\epsilon \rightarrow 0+$ .

This kind of approximation can also be found in  $s$ -wave superconductors. The conventional  $s$ -wave Ginzburg-Landau free energy (cf. [9]) without the magnetic field is

$$\int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2,$$

where  $u \in \mathbb{C}$  is the  $s$ -wave order parameter. Under the hypothesis that  $|u| \rightarrow 1$  and all the derivatives of  $u$  decay fast at  $|(x, y)| \rightarrow \infty$ , we may approximate the  $s$ -wave Ginzburg-Landau free energy by

$$\int_{\frac{1}{\epsilon}\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2,$$

where  $0 < \epsilon \ll 1$  is a small parameter and  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^2$  having an interior point at the origin. Then we rescale the spatial variables by  $\epsilon$  and obtain the energy functional as follows:

$$E_\epsilon(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4\epsilon^2} (1 - |u|^2)^2, \quad (2.4)$$

where  $u : \Omega \rightarrow \mathbb{C}$  is the  $s$ -wave order parameter. There are many investigations on the free energy (2.4). For the readers who are interested in these works, please refer to [2], [15], [17], [23] and [29] etc.

In [2] and [29], we learn the minimizer of  $E_\epsilon$  over  $H_g^1(\Omega)$  having  $n$  degree-one vortices in  $\Omega$ , where

$$H_g^1(\Omega) = \{u \in H^1(\Omega; \mathbb{C}) : u = g \quad \text{on } \partial\Omega\},$$

and  $g : \partial\Omega \rightarrow S^1$  is smooth with degree  $n \geq 1$ . Furthermore, the minimizer  $u_\epsilon$  of (2.4) satisfies

- (1)  $E_\epsilon(u_\epsilon) = n\pi \log \frac{1}{\epsilon} + W_g(a_1, \dots, a_n) + o_\epsilon(1)$  as  $\epsilon \rightarrow 0+$ ,
- (2)  $u_\epsilon$  converges to  $u_*$  (up to a subsequence) in  $C_{loc}^2(\bar{\Omega} \setminus \{a_1, \dots, a_n\})$  as  $\epsilon \rightarrow 0+$ ,
- (3)  $(a_1, \dots, a_n) \in \Omega^n$  is a global minimizer of the renormalized energy  $W_g$  defined in [2],

where  $o_\epsilon(1)$  is a small quantity which tends to zero as  $\epsilon \rightarrow 0+$ ,

$$u_*(z) = \prod_{j=1}^n \frac{z - a_j}{|z - a_j|} e^{i h(z)}, \quad \forall z \in \Omega, \quad (2.5)$$

and  $h$  is a real-valued harmonic function. Since  $\mathbb{R}^2$  is isomorphic to  $\mathbb{C}$ , we may consider  $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ . Note that the domain  $\Omega$  is assumed star-shaped in [2]. However, Struwe [29] generalized results of [2] for all bounded smooth domains.

For the minimizer of (2.2), we prove:

**Theorem I.** *Suppose  $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$  as  $\epsilon \rightarrow 0+$ . Then there exists a minimizer  $u_\epsilon$  of (2.2) over  $H_g^1(\Omega)$  such that*

- (i)  $u_\epsilon \in H^2(\Omega)$  has  $n$  degree-one vortices in  $\Omega$ ,
- (ii)  $\hat{G}_\epsilon(u_\epsilon) = 2n\pi \log \frac{1}{\epsilon} + O(1)$  as  $\epsilon \rightarrow 0+$ ,
- (iii)  $u_\epsilon$  converges to  $u_*$  (up to a subsequence) strongly in  $L^2(\Omega)$  and weakly in  $H_{loc}^1(\Omega \setminus \{a_1, \dots, a_n\})$ ,
- (iv)  $(a_1, \dots, a_n) \in \Omega^n$  is a global minimizer of the renormalized energy  $W_g$  in [2].

**Remark.** We may consider the energy functional (2.2) with  $0 < \delta_\epsilon, \gamma_\epsilon = O(\epsilon^2)$  as a small perturbation of (2.4). However, the perturbation terms are of higher order derivatives. Hence the Euler-Lagrange equation of (2.2) is a singular perturbation problem and the perturbation terms are of the 4-th order derivatives. Until now, there is no general theorem on such a singular perturbation problem.

### 3 Proof of Theorem I

To prove the existence of a minimizer, we define a comparison map as follows:

$$U_\epsilon(z) = \prod_{j=1}^n U_0\left(\frac{z - b_j}{\epsilon}\right) e^{i H_\epsilon(z)}, \quad (3.1)$$

for  $z \in \Omega \subset \mathbb{C}$ , where  $b_j$ 's are  $n$  distinct points in  $\Omega$  and  $H_\epsilon$  is a real-valued smooth function in  $\Omega$  such that

$$U_\epsilon = g \quad \text{on } \partial\Omega, \quad \|H_\epsilon\|_{C^2(\Omega)} = O(1).$$

Hereafter,  $U_0$  is the symmetric vortex solution (cf. [4], [10], [11]) defined by

$$U_0(z) = f(R) e^{i\theta} \quad \text{for } z \in \mathbb{C}, \quad (3.2)$$

where  $R = |z|$  and  $(R, \theta)$  is the polar coordinate in  $\mathbb{C}$ . Moreover,  $f(R)$  satisfies

$$\begin{cases} f'' + \frac{1}{R} f' - \frac{1}{R^2} f + (1 - f^2)f = 0 & \text{for } R > 0, \\ f(0) = 0, f(\infty) = 1. \end{cases} \quad (3.3)$$

From [4] and [11], the symmetric vortex solution  $U_0$  satisfies

**Lemma I.**

- (i)  $f(R) = \alpha_0 R + \alpha_1 R^3 + O(R^5)$  as  $R \rightarrow 0+$ , where  $\alpha_0 > 0, \alpha_1 \in \mathbb{R}$  are constants,
- (ii)  $f(R) = 1 - \frac{1}{2R^2} + O(R^{-4})$  as  $R \rightarrow +\infty$ ,
- (iii)  $U_0 = f(R) e^{i\theta}$  is analytic in  $\mathbb{C}$ .

Hence it is easy to check that

$$\hat{G}_\epsilon(U_\epsilon) = 2\pi n \log \frac{1}{\epsilon} + O(1) \quad \text{as } \epsilon \rightarrow 0+. \quad (3.4)$$

Now, fix  $0 < \epsilon \ll 1$ . We claim that  $\inf_{u \in H_g^1(\Omega)} \hat{G}_\epsilon(u)$  attains a minimizer  $u_\epsilon \in H^2(\Omega)$ . Let  $\{u_k\}$  be a minimizing sequence such that

$$\hat{G}_\epsilon(u_k) \rightarrow \inf_{u \in H_g^1(\Omega)} \hat{G}_\epsilon(u). \quad (3.5)$$

Then by (2.2), (3.4) and (3.5), we have

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^2 + |\square u_k|^2 + |\Delta u_k|^2 dx dy < +\infty.$$

Hence there exists a subsequence  $\{u_{k_j}\}$  such that

$$\|u_{k_j}\|_{H^2} \leq K_\epsilon, \quad \forall j \geq 1, \quad (3.6)$$

where  $K_\epsilon > 0$  is a constant independent of  $j$ . Thus (3.6) implies

$$u_{k_j} \rightarrow u_\epsilon \quad \text{weakly in } H^2(\Omega) \quad \text{as } j \rightarrow \infty. \quad (3.7)$$

Therefore by Fatou's lemma,  $u_\epsilon$  is a minimizer of  $\hat{G}_\epsilon$  over  $H_g^1(\Omega)$ .

From (2.2), (2.4), (3.4) and  $u_\epsilon$  is a minimizer of  $\inf_{u \in H_g^1(\Omega)} \hat{G}_\epsilon(u)$ , we obtain

$$E_\epsilon(u_\epsilon) \leq \pi n \log \frac{1}{\epsilon} + O(1). \quad (3.8)$$

Moreover, by (3.8) and [29], we have

$$E_\epsilon(u_\epsilon) = \pi n \log \frac{1}{\epsilon} + O(1). \quad (3.9)$$

Hence (3.4) and (3.9) imply that

$$\delta_\epsilon \int_\Omega |\square u_\epsilon|^2 dx dy = O(1), \quad (3.10)$$

and

$$\gamma_\epsilon \int_\Omega |\Delta u_\epsilon|^2 dx dy = O(1). \quad (3.11)$$

Thus we complete the proof of (ii).

By (3.9), Proposition 1.1 and 1.2 in [16], we complete the proof of (i). Furthermore, we obtain that  $u_\epsilon$  converges to  $u_*$  (up to a subsequence) strongly in  $L^2(\Omega)$  and weakly in  $H_{loc}^1(\Omega \setminus \{a_1, \dots, a_n\})$ , where  $a_1, \dots, a_n \in \Omega$ ,  $u_*(z) = \prod_{j=1}^n \frac{z - a_j}{|z - a_j|} e^{i h(z)}$ ,  $\forall z \in \Omega \subset \mathbb{C}$  and  $h$  is a real-valued function. Now we show that  $h$  is a harmonic function as follows: Consider the Euler-Lagrange equation of  $\hat{G}_\epsilon$  with respect to the minimizer  $u_\epsilon$ . Then  $u_\epsilon$  satisfies

$$\Delta u_\epsilon + \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2) u_\epsilon - \delta_\epsilon \square^2 u_\epsilon - \gamma_\epsilon \Delta^2 u_\epsilon = 0 \quad \text{in } \Omega. \quad (3.12)$$

Perform the wedge product with  $u_\epsilon$  and (3.12). This is a standard trick to erase the cubic nonlinear term in (3.12) (cf. [26] and [29]). Then we have

$$u_\epsilon \wedge \Delta u_\epsilon - \delta_\epsilon u_\epsilon \wedge \square^2 u_\epsilon - \gamma_\epsilon u_\epsilon \wedge \Delta^2 u_\epsilon = 0 \quad \text{in } \Omega. \quad (3.13)$$

Let  $p \in C_0^\infty(\Omega)$  be a test function. Multiply (3.13) by  $p$  and integrate it on  $\Omega$ . Then using integration by parts, we obtain

$$\begin{aligned} & - \int_\Omega (u_\epsilon \wedge \partial_x u_\epsilon) p_x + (u_\epsilon \wedge \partial_y u_\epsilon) p_y \\ & = \delta_\epsilon \int_\Omega (u_\epsilon \wedge \square u_\epsilon) \square p + 2(\partial_x u_\epsilon \wedge \square u_\epsilon) p_x - 2(\partial_y u_\epsilon \wedge \square u_\epsilon) p_y \\ & \quad + \gamma_\epsilon \int_\Omega (u_\epsilon \wedge \Delta u_\epsilon) \Delta p + 2(\partial_x u_\epsilon \wedge \Delta u_\epsilon) p_x + 2(\partial_y u_\epsilon \wedge \Delta u_\epsilon) p_y \end{aligned} \quad (3.14)$$

Here we have used the following formulas:

$$\begin{aligned} u \wedge \Delta u &= \partial_x (u \wedge \partial_x u) + \partial_y (u \wedge \partial_y u), \\ u \wedge \square^2 u &= \square(u \wedge \square u) - 2(u_x \wedge \square u_x - u_y \wedge \square u_y), \\ u \wedge \Delta^2 u &= \Delta(u \wedge \Delta u) - 2(u_x \wedge \Delta u_x + u_y \wedge \Delta u_y). \end{aligned}$$

Hence by  $0 < \gamma_\epsilon, \delta_\epsilon = O(\epsilon^2)$ , (3.9)-(3.11), (3.14) and Holder inequality, the limit map  $u_*$  satisfies

$$u_* \wedge \Delta u_* = 0 \quad \text{in distribution sense.} \quad (3.15)$$



Thus  $u_*$  is a canonical harmonic map i.e.  $h$  is a harmonic function. Therefore we complete the proof of (iii).

Now we prove (iv) as follows: Let  $(\tilde{a}_1, \dots, \tilde{a}_n) \in \Omega^n$  be a global minimizer of the renormalized energy  $W_g$ . The definition of  $W_g$  can be found in [2]. Then we define another comparison map as follows:

$$V_\epsilon(z) = \begin{cases} u_\epsilon(z - \tilde{a}_j + a_j) & \text{if } z \in B_{\epsilon^\alpha}(\tilde{a}_j), j = 1, \dots, n, \\ \tilde{U}_\epsilon(z) & \text{if } z \in \Omega_{\epsilon^\alpha} \equiv \Omega \setminus \bigcup_{j=1}^n B_{\epsilon^\alpha}(\tilde{a}_j), \end{cases} \quad (3.16)$$

where  $0 < \alpha < 1$  is a constant and  $\tilde{U}_\epsilon$  is a minimizer of  $E_\epsilon$  over  $H_g^1(\Omega_{\epsilon^\alpha})$ . Here the boundary condition  $\tilde{g}$  is defined by

$$\tilde{g} = \begin{cases} g & \text{on } \partial\Omega, \\ u_\epsilon(\cdot - \tilde{a}_j + a_j) & \text{on } \partial B_{\epsilon^\alpha}(\tilde{a}_j), j = 1, \dots, n. \end{cases} \quad (3.17)$$

Hence by (iii), [2] and [29],  $\tilde{U}_\epsilon$  satisfies

$$\tilde{U}_\epsilon \rightarrow \prod_{j=1}^n \frac{z - \tilde{a}_j}{|z - \tilde{a}_j|} e^{i\tilde{h}(z)} \quad \text{in } C^2(\Omega_{\epsilon^\alpha}) \quad \text{as } \epsilon \rightarrow 0+, \quad (3.18)$$

where  $\tilde{h}$  is a harmonic function. The convergence of (3.18) may be up to a subsequence. However, this does not effect the following argument. Thus by (3.18) and [2], it is easy to check that

$$\hat{G}_\epsilon(V_\epsilon) = \sum_{j=1}^n \int_{B_{\epsilon^\alpha}(a_j)} \hat{g}_\epsilon(u_\epsilon) + 2\pi n \alpha \log \frac{1}{\epsilon} + 2W_g(\tilde{a}_1, \dots, \tilde{a}_n) + o_\epsilon(1), \quad (3.19)$$

where  $\hat{g}_\epsilon(u) = |\nabla u|^2 + \frac{1}{2\epsilon^2}(1 - |u|^2)^2 + \delta_\epsilon |\square u|^2 + \gamma_\epsilon |\Delta u|^2$  is the energy density of  $\hat{G}_\epsilon$  and  $o_\epsilon(1)$  is a small quantity which tends to zero as  $\epsilon \rightarrow 0+$ . On the other hand, by (iii) and [2], we have

$$\int_{\hat{\Omega}_{\epsilon^\alpha}} \frac{1}{2} |\nabla u_\epsilon|^2 + \frac{1}{4\epsilon^2} (1 - |u_\epsilon|^2)^2 \geq \pi n \alpha \log \frac{1}{\epsilon} + W_g(a_1, \dots, a_n) + o_\epsilon(1), \quad (3.20)$$

where  $\hat{\Omega}_{\epsilon^\alpha} = \Omega \setminus \bigcup_{j=1}^n B_{\epsilon^\alpha}(a_j)$ . Hence (3.20) implies that

$$\hat{G}_\epsilon(u_\epsilon) \geq \sum_{j=1}^n \int_{B_{\epsilon^\alpha}(a_j)} \hat{g}_\epsilon(u_\epsilon) + 2\pi n \alpha \log \frac{1}{\epsilon} + 2W_g(a_1, \dots, a_n) + o_\epsilon(1), \quad (3.21)$$

Thus by (3.19) and (3.21), we obtain

$$W_g(a_1, \dots, a_n) \leq W_g(\tilde{a}_1, \dots, \tilde{a}_n) + o_\epsilon(1) \quad (3.22)$$

Since  $(\tilde{a}_1, \dots, \tilde{a}_n)$  is a global minimizer of  $W_g$ , then we complete the proof of (iv) by (3.22).

## 4 Single Vortex Structure in the Vortex Core

In this section, we assume that the single vortex structure is in the vortex core  $B_{R_0}$ , where  $R_0 > 0$  is a large constant satisfying  $1 \ll R_0 \leq \frac{1}{\epsilon}$ . Hereafter, we denote  $B_{R_0}$  as a disk in  $\mathbb{R}^2$  with radius  $R_0$  and center at the origin. To study the vortex structure in the vortex core, we restrict (1.9) in the vortex core  $B_{R_0}$  as follows:

$$\hat{G}(\Psi_d) = \int_{B_{R_0}} |\nabla \Psi_d|^2 + \frac{1}{2}(1 - |\Psi_d|^2)^2 + \beta |\square \Psi_d|^2 + \gamma |\Delta \Psi_d|^2 dx dy, \quad (4.1)$$

where  $\gamma = C\beta$ ,  $C > 0$  is a constant independent of  $\beta$ , and  $\beta > 0$  is a small parameter tending to zero as  $T \rightarrow T_c$ . We investigate (4.1) with  $\beta > 0$  a small parameter to see the phase transition of  $d$ -wave superconductors.

The Euler-Lagrange equation of (4.1) is

$$\Delta \Psi_d + (1 - |\Psi_d|^2)\Psi_d - \beta(\square^2 + C\Delta^2)\Psi_d = 0 \quad \text{in } B_{R_0}. \quad (4.2)$$

Note that  $E \equiv \square^2 + C\Delta^2$  is an elliptic operator as  $C > 0$ . Moreover, by the Lax-Milgram theorem,  $E : H_0^2(B_{R_0}; \mathbb{C}) \rightarrow H^{-2}(B_{R_0}; \mathbb{C})$  is invertible and we denote  $E^{-1}$  as its inverse. Hence the standard elliptic regularity theorem (cf. [20]) can be applied in (4.2).

We state the main result on (4.2) as follows:

**Theorem II.** *There exists a solution  $\Psi_d$  of (4.2) satisfying*

$$\Psi_d(z, \beta) = f(R) e^{i\theta} + \beta(a(R) e^{-4i\theta} + b(R) e^{4i\theta} + c(R)) e^{i\theta} + O(\beta^2) \quad \text{as } \beta \rightarrow 0, \quad (4.3)$$

where  $a, b$  and  $c$  are smooth real-valued functions.

The equation (4.3) implies that the  $d$ -wave order parameter  $\Psi_d$  is fourfold symmetric in the vortex core. In [27], we learn a well approximated solution of (4.2) with fourfold symmetry. Here we find an exact solution of (4.2) with the fourfold symmetry.

**Proof of Theorem II.**

To solve (4.2), we set

$$\Psi_d(z, \beta) = U_0(z) + \beta w_1(z) + \beta^2 w_2(z) + \beta^3 w(z, \beta), \quad (4.4)$$

where  $U_0$  is the symmetric vortex solution defined in (3.2) and (3.3). Here  $w_1$  satisfies

$$L w_1 - E U_0 = 0 \quad \text{in } B_{R_0}, \quad w_1 = 0 \quad \text{on } \partial B_{R_0}, \quad (4.5)$$

where  $Lv = \Delta v + (1 - |U_0|^2)v - 2(U_0 \cdot v)U_0$  is the linearized operator of the equation (4.2) with respect to a trivial solution  $(\Psi_d, \beta) = (U_0, 0)$ . In addition,  $w_2$  satisfies that

$$\begin{aligned} L w_2 &= 2(U_0 \cdot w_1)w_1 + |w_1|^2 U_0 + E w_1 \quad \text{in } B_{R_0}, \\ w_2 &= 0 \quad \text{on } \partial B_{R_0}. \end{aligned} \quad (4.6)$$

It is easy to check that

$$EU_0 = h_{-3}(R) e^{-3i\theta} + h_1(R) e^{i\theta} + h_5(R) e^{5i\theta}, \quad (4.7)$$

where  $h_{-3}, h_1$  and  $h_5$  are real-valued smooth functions. By [14], [18], [19] and [23],  $L$  is a bijection from  $H_0^1(B_{R_0}; \mathbb{C})$  onto  $H^{-1}(B_{R_0}; \mathbb{C})$ . Hence by (4.5)-(4.7), we have

$$w_1 = a(R) e^{-3i\theta} + b(R) e^{5i\theta} + c(R) e^{i\theta}, \quad (4.8)$$

$$w_2 = \sum_{k=0}^2 a_{1-4k}(R) e^{i(1-4k)\theta} + a_{1+4k}(R) e^{i(1+4k)\theta}, \quad (4.9)$$

where  $a, b, c$  and  $a_{1\pm 4k}$ 's are smooth real-valued functions.

Taking (4.4) into (4.2), we obtain that

$$\begin{aligned} Lw = & 2[(U_0 \cdot (w_2 + \beta w))w_1 + (U_0 \cdot w_1)(w_2 + \beta w)] + \beta|w_2 + \beta w|^2 U_0 \\ & + \beta(U_0 \cdot (w_2 + \beta w))(w_2 + \beta w) + 2(w_1 \cdot (w_2 + \beta w))U_0 \\ & + |w_1 + \beta(w_2 + \beta w)|^2(w_1 + \beta(w_2 + \beta w)) + Ew_2 + \beta Ew \quad \text{in } B_{R_0}. \end{aligned} \quad (4.10)$$

Hence (4.10) is equivalent to

$$\begin{aligned} E^{-1}Lw = & E^{-1}\{2[(U_0 \cdot (w_2 + \beta w))w_1 + (U_0 \cdot w_1)(w_2 + \beta w)] + \beta|w_2 + \beta w|^2 U_0 \\ & + \beta(U_0 \cdot (w_2 + \beta w))(w_2 + \beta w) + 2(w_1 \cdot (w_2 + \beta w))U_0 \\ & + |w_1 + \beta(w_2 + \beta w)|^2(w_1 + \beta(w_2 + \beta w))\} + w_2 + \beta w \quad \text{in } B_{R_0}. \end{aligned} \quad (4.11)$$

Note that (4.11) has a trivial solution  $(w, \beta) = (w_3, 0)$ , where  $w_3$  satisfies that

$$\begin{aligned} Lw_3 &= 2[(U_0 \cdot w_2)w_1 + (U_0 \cdot w_1)w_2] + 2(w_1 \cdot w_2)U_0 + |w_1|^2 w_1 + Ew_2 \quad \text{in } B_{R_0}, \\ w_3 &= 0 \quad \text{on } \partial B_{R_0}. \end{aligned} \quad (4.12)$$

Since  $U_0, w_1, w_2$  are smooth functions and  $L$  is bijective from  $H_0^1(B_{R_0}; \mathbb{C})$  onto  $H^{-1}(B_{R_0}; \mathbb{C})$ , then the standard elliptic regularity theorem implies that  $w_3$  is also a smooth function. Furthermore, since  $E$  is bijective from  $H_0^2(B_{R_0}; \mathbb{C})$  onto  $H^{-2}(B_{R_0}; \mathbb{C})$  and  $H^{-1}(B_{R_0}; \mathbb{C})$  is embedded in  $H^{-2}(B_{R_0}; \mathbb{C})$ , then  $E$  is a bijection from  $H_0^2(B_{R_0}; \mathbb{C}) \cap H^3(B_{R_0}; \mathbb{C})$  onto  $H^{-1}(B_{R_0}; \mathbb{C})$ . We denote  $E^{-1}$  as the inverse of  $E$ . Hence  $E^{-1}L$  is a bijection from  $H_0^1(B_{R_0}; \mathbb{C})$  onto  $H_0^2(B_{R_0}; \mathbb{C}) \cap H^3(B_{R_0}; \mathbb{C})$ . Thus by the implicit function theorem, (4.11) has a unique solution  $w \in H_0^1(B_{R_0}; \mathbb{C})$  as  $|\beta|$  is sufficiently small. Moreover, the standard elliptic regularity theorem may imply the smoothness of  $w$ . Therefore (4.2) has a solution  $\Psi_d$  satisfying (4.4) as  $|\beta|$  is sufficiently small. By (4.4), (4.8) and (4.9), we obtain (4.3) and complete the proof of Theorem II.

**Final Remark:** By (1.4) with  $A \equiv 0$  and (4.3), we have

$$\Psi_s(z) = \xi \square [U_0 + \beta(a(R) e^{-4i\theta} + b(R) e^{4i\theta} + c(R)) e^{i\theta} + O(\beta^2)] \quad \text{as } \beta \rightarrow 0. \quad (4.13)$$

Since  $U_0(z) = f(R) e^{i\theta}$ , then

$$\square U_0(z) = \frac{1}{2}(f' + \frac{1}{R}f)' e^{-i\theta} + \frac{1}{2}[(f' - \frac{1}{R}f)' - \frac{2}{R}(f' - \frac{1}{R}f)] e^{3i\theta}. \quad (4.14)$$

Hence by (i), (ii) of Lemma I and (4.14),  $\square U_0$  satisfies

$$\square U_0(z) = 4\alpha_1 R e^{-i\theta} + O(R^3) \quad \text{as } R \rightarrow 0+, \quad (4.15)$$

and

$$\square U_0(z) = -\frac{1}{2R^2} e^{-i\theta} + \frac{3}{2R^2} e^{3i\theta} + O(R^{-4}) \quad \text{as } R \rightarrow +\infty. \quad (4.16)$$

By (4.15) and (4.16), the degree of  $\square U_0$  is minus one in  $B_{r_1}$  and three in  $B_{R_1}$  as  $0 < r_1 \ll 1$  and  $R_1 \gg 1$ . Moreover, by [4] and [11], it is easy to check that

$$\frac{d}{dz} \square U_0(z) \neq 0 \quad \text{if } \square U_0(z) = 0. \quad (4.17)$$

Hence (iii) of Lemma I and (4.17) imply that  $\square U_0$  has only simple zeros in  $\mathbb{C}$ . Thus  $\square U_0$  has a single zero with degree minus one at the origin and another four zeros with degree one away from the origin. Therefore as  $|\beta|$  is sufficiently small,  $\Psi_s$  has a single zero with degree minus one near the origin and another four zeros with degree one away from the origin. This indicates the four-lobe structure of  $\Psi_s$  in the vortex core. The numerical simulation can be found in [7], [8] and [31].

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